

Yang–Mills Fields as Optical Media

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A geometrization of the Yang–Mills field, by which an $SU(2)$ gauge theory becomes equivalent to a 3-space geometry—or optical system—is examined. In a first step, ambient space remains Euclidean and current problems on flat space can be looked at from a new point of view. The Wu–Yang ambiguity, for example, appears related to the multiple possible torsions of distinct metric-preserving connections. In a second step, the ambient space also becomes curved. In the generic case, the strictly Riemannian metric sector plays the role of an arbitrary host space, with the gauge potential represented by a contorsion. For some field configurations, however, it is possible to obtain a purely metric representation. In those cases, if the space is symmetric homogeneous, the Christoffel connections are automatically solutions of the Yang–Mills equations.

1. INTRODUCTION

Our main intuitive guide to interactions is, ultimately, the nonrelativistic idea of potential. We recall, for example, the phenomenological potentials used with reasonable success in low-energy hadron spectroscopy: a Coulomb-like term, plus a linear potential providing for the confining behavior, are thought to represent the nonrelativistic limit of the time components A^a_0 of the gauge potential in chromodynamics. The Wilson-loop criterion for confinement gives potentials of that kind in the nonrelativistic limit, and is thereby justified. There are, however, great advantages in the use of the (temporal, or Weyl) gauge $A^a_0 = 0$, which is obviously incompatible with such a view. The merits of this gauge (Feynman, 1977) are particularly relevant when associated to the Hamiltonian formalism (Jackiw, 1980). The major qualitative characteristics of gauge fields, such as shielding and confinement, are nowadays believed to be essentially nonperturbative, and the best approach available to consider global aspects is precisely the Hamiltonian

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formalism. On the other hand, the impossibility of thinking in terms of a potential inhibits intuition. It would be nice to have some other qualitative guide in its stead. This paper is intended to call attention to the possibility of—at least in some cases—using a geometrical optics analogy. A static sourceless $SU(2)$ gauge field configuration can, in the temporal gauge, be “geometrized” to become equivalent to a metric plus a torsion on 3-space. A metric on a 3-space is a simple optical system (Guillemin and Sternberg, 1977), as it can be seen as the dielectric tensor ϵ_{ij} of a medium to which torsion will add some defects (Aldrovandi and Pereira, 1995). An optical picture, with refractive indices and defects taking the place of potentials, could be an alternative source of ideas.

Geometrization of the $SU(2)$ theory was proposed by Lunev (1992), and by Freedman *et al.* (1993; see also Haagensen, 1993, and Johnson, 1992) in a tentative to arrive at a description of gauge fields in terms of invariants. In particular, it was part of a program (Freedman *et al.*, 1993) to solve a great problem in the quantization of the Hamiltonian scheme—the implementation of Gauss’ law. In that pursuit some assumptions were made which were not necessary to the simpler aim of establishing a geometrized version of the theory. We present in the following a minimal approach, using only the hypotheses strictly necessary to that particular end. It turns out that, in the generic case, the metric sector is highly arbitrary and acts as a “host” space, on which the “guest” gauge potential is represented by the contorsion tensor. For some field configurations, however, it is possible to choose a metric which alone contains all the information.

We begin by recalling the main aspects of the Hamiltonian approach to Yang–Mills theory, in which time and Euclidean 3-space \mathbf{E}^3 are clearly separated. We show then how to transcribe the field equations into those of a geometry on \mathbf{R}^3 . Complete geometrization would lead immediately to gauge theories on curved spaces. We proceed consequently in two steps. In the first, only the indices related to the Lie algebra are “geometrized,” while ambient space remains the flat space \mathbf{E}^3 . The main geometrical ideas are already present, but we remain able to discuss questions turning up in flat space. Some of them are seen under a new angle. For example, the Wu–Yang ambiguity is related to the multiplicity of torsion tensors with a fixed curvature. In the second step, also the ambient \mathbf{R}^3 is endowed with a new, rather arbitrary metric. The Yang–Mills equations appear then written on curved spaces. If such spaces are torsionless homogeneous symmetric spaces, their very Christoffel connections are solutions. In such cases, the full geometrization exhibits field configurations which are completely equivalent to simple optical systems. The group really considered is $SU(2)$, for which the geometry unfolds itself in a quite natural way. In our notation, Greek indices run from 0 to 3 and Latin indices from 1 to 3.

2. HAMILTONIAN FORMALISM

In the Hamiltonian approach (Faddeev and Slavnov, 1978; Itzykson and Zuber, 1980; Ramond, 1981) to the Yang–Mills equations, the canonical coordinates are the vector potential components A^a_k and, once the Lagrangian $\mathcal{L} = -\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a$ is given, the conjugate momenta Π^a_k are the electric fields:

$$\Pi^{ai} = \frac{\delta\mathcal{L}}{\delta\partial_0 A^a_i} = F^{ai0} = E^{ai} = \partial^i A^{a0} - \partial^0 A^{ai} + f^a_{bc} A^{bi} A^{c0} \quad (1)$$

The action can then be rewritten in the form

$$S = 2 \int d^4x \operatorname{tr} \left[\partial^0 \mathbf{A} \cdot \mathbf{E} + \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) - A_a^0 G^a(x) \right] \quad (2)$$

where

$$G^a(x) = D_k E^{ak} = \partial_k E^{ak} + f^a_{bc} A^b_k E^{ck} \quad (3)$$

(we have profited to define the derivative D_k). A constraint, a redefinition of terms, and two dynamic equations come out. The first two are as follows:

(i) The Gauss law, which states the vanishing of (3):

$$G^a(x) = \partial_k E^{ak} + f^a_{bc} A^b_k E^{ck} = 0 \quad (4)$$

[where we see that A_0 is, in action (2), a Lagrange multiplier enforcing Gauss' law].

(ii) The expression of the magnetic field in terms of A^a_k ,

$$B^{ai} = \frac{1}{2} \epsilon^{ijk} F^a_{jk} = \epsilon^{ijk} (\partial_j A^a_k - \partial_k A^a_j + \frac{1}{2} f^a_{bc} A^b_j A^c_k) \quad (5)$$

The dynamic equations are Hamilton's equations:

(iii) The time variation of the vector potential,

$$\frac{1}{c} \frac{\partial}{\partial t} A^a_i = -E^a_i - \partial_i A^{a0} + f^a_{bc} A^{b0} A^c_i \quad (6)$$

(iv) Ampère's law, here in the role of the force law:

$$\frac{1}{c} \frac{\partial}{\partial t} E^{ai} = (\nabla \times \mathbf{B})^{ai} + \epsilon^i_{jk} f^a_{bc} A^{bj} B^{ck} + f^a_{bc} A^{b0} E^{ci} \quad (7)$$

Consider now the gauge $A^a_0 = 0$. The Hamiltonian is

$$H = \frac{1}{2} \int d^3x \operatorname{tr}[(\mathbf{E}^2 + \mathbf{B}^2)] \quad (8)$$

A static A^a_i leads to $E^a_i = 0$. Gauss' law is automatically satisfied and Ampère's law reduces to

$$(\nabla \times \mathbf{B})^{ai} + \epsilon^{ijk} f^a{}_{bc} A^{bj} B^{ck} = 0 \quad (9)$$

Notice that (5) fixes B once A is given, but not vice versa. This is the Wu–Yang ambiguity (Wu and Yang, 1975): many inequivalent gauge fields $A^a{}_i$ can correspond to the same magnetic field $B^a{}_i$. A consequence is that the $B^a{}_i$ cannot be used as coordinates.

3. STEPPING INTO GEOMETRY

Let us stress beforehand that many different metrics can be defined on the same space (for us, “space” will mean only a differentiable manifold). The best examples of such a metric multiplicity are provided precisely by optical systems (Luneburg, 1966), whose treatment is greatly eased by the simultaneous use of the Euclidean metric δ_{ij} of \mathbf{E}^3 and of the dielectric tensor ϵ_{ij} . Isotropic media have $\epsilon_{ij} = n^2 \delta_{ij}$, with n the refractive index and correspond to conformally flat 3-spaces. Notice that by \mathbf{E}^3 we understand the usual Euclidean metric space, the space \mathbf{R}^3 of real ordered triples endowed with its unique differentiable structure and with the additional proviso that length measurements are performed supposing that $ds^2 = \delta_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2$. An optical system will be the same differentiable manifold \mathbf{R}^3 , but with optical lengths instead, measured with the dielectric metric $dl^2 = \epsilon_{ij} dx^i dx^j$. Another example is given by the group $SU(2)$, whose manifold is the 3-sphere S^3 . It has the “natural” spherical metric which comes up when S^3 is seen as an imbedded submanifold of the Euclidean space \mathbf{E}^4 , but it has also the Killing–Cartan metric $\gamma_{ab} = \delta_{ab}$, which is more important from the algebraic point of view.

Now, looking back at what comes out in the Hamiltonian formalism: The only equations remaining in the static case are (5) and (9). They can be easily rewritten in terms of a spatial geometry in the following way. First, we notice that both the spaces involved are 3-dimensional, on which vectors are equivalent to antisymmetric 2-tensors. Indices can be trivially “dualized.” We can redefine the gauge potential as the connection

$$\omega^a{}_{ck} = \epsilon^a{}_{bc} A^b{}_k \quad (10)$$

with curvature

$$R^a{}_{bij} = \epsilon^a{}_{cb} F^c{}_{ij} = \epsilon_{ijk} \epsilon^a{}_{cb} B^{ck} \quad (11)$$

in terms of which (5) becomes

$$R^a{}_{bij} = \partial_i \omega^a{}_{bj} - \partial_j \omega^a{}_{bi} + \omega^a{}_{ci} \omega^c{}_{bj} - \omega^a{}_{cj} \omega^c{}_{bi} \quad (12)$$

We find immediately that, for any object of index type W^a ,

$$[D_i, D_j]W^a = R^a{}_{bij}W^b \tag{13}$$

Second, we notice that the tangent space to the sphere $SU(2)$, with the metric γ_{ab} , is isomorphic to \mathbf{E}^3 and, consequently, to the space tangent to the ambient space on which the differential equations are written. There are actually infinite such isomorphisms, each one realized by a dreibein field $h^a{}_i$ (we shall be using the letters a, b, c, \dots as isotopic spin indices and i, j, r, s, \dots as ambient space indices). A dreibein field, together with its inverse $h^i{}_b$, can be used to change tensor indices as in

$$R^r{}_{sij} = h^r{}_a h^b{}_s R^a{}_{bij} \tag{14}$$

Connections, however, are not truly tensorial. Only the last (in our notation), derivative index is a covector index. The other are not, and are translated according to

$$\Gamma^i{}_{jk} = h^i{}_a \omega^a{}_{bk} h^b{}_j + h^i{}_c \partial_k h^c{}_j \tag{15}$$

This comes from the requirement that the covariant derivative remain covariant under change of basis. The connection $\omega^a{}_{bk}$ would represent, in the absence of torsion, the Ricci rotation coefficients (Chandrasekhar 1992) or, if we borrow from the usual treatment of the Dirac equation on curved spaces, the spin connection. With the above transformations, equations (5) and (9) become

$$R^r{}_{sij} = \partial_i \Gamma^r{}_{sj} - \partial_j \Gamma^r{}_{si} + \Gamma^r{}_{ki} \Gamma^k{}_{sj} - \Gamma^r{}_{kj} \Gamma^k{}_{si} \tag{16}$$

and

$$\partial_j R^r{}_{si}{}^j + \Gamma^r{}_{kj} R^k{}_{si}{}^j - \Gamma^k{}_{sj} R^r{}_{ki}{}^j = 0 \tag{17}$$

where now all the indices refer to the ambient space. Notice that the dreibeine are quite arbitrary. Equation (16) simply defines $R^r{}_{sij}$ as the curvature of the connection $\Gamma^r{}_{si}$, but the nine equations (17), stating Ampère’s law, keep their dynamical role.

Another characteristic of the connection, its torsion, will be given by

$$T^a{}_{ij} = \partial_i h^a{}_j - \partial_j h^a{}_i + \omega^a{}_{ci} h^c{}_j - \omega^a{}_{cj} h^c{}_i \tag{18}$$

or, after transmuting the indices,

$$T^k{}_{ij} = -\Gamma^k{}_{[ij]} \tag{19}$$

We are introducing the notation $[ij]$ for antisymmetrized indices without any numerical factors, and we shall use (ij) for symmetrization. This will lead, for example, to the identity

$$\Gamma^k{}_{ij} = \frac{1}{2}(\Gamma^k{}_{(ij)} + \Gamma^k{}_{[ij]}) \tag{20}$$

Some formal expressions are of interest to ease manipulations: first,

$$\Gamma^k_{ij} = h_a{}^k D_j h^a{}_i \quad (21)$$

and its consequence

$$T^k_{ij} = h_a{}^k D_{[i} h^a{}_{j]} \quad (22)$$

Equation (13) implies

$$R^r{}_{sij} = h_a{}^r [D_i, D_j] h^a{}_s \quad (23)$$

Finally, the Bianchi identity for torsion,

$$D_{[i} T^a{}_{jk]} = R^a{}_{[ijk]} \quad (24)$$

The dreibein field will define a metric on \mathbf{R}^3 by

$$g_{ij} = \gamma_{ab} h^a{}_i h^b{}_j \quad (25)$$

This metric is automatically preserved by Γ^k_{ij} . Indeed, the metric compatibility condition (which means that the metric is parallel-transported by Γ)

$$\partial_k g_{ij} = \Gamma_{ijk} + \Gamma_{jik} = \Gamma_{(ij)k} \quad (26)$$

is a simple consequence of (15).

Thus, in the transcription of Yang–Mills fields into a spatial geometry induced by a dreibein field, the gauge potential is transmuted into a connection which automatically preserves the metric defined by the dreibeine. Given a dreibein field and the original A , the connection Γ is unique. This is due to the Ricci lemma (Greub, 1972), which reads: Given a metric g and any tensor of type T^k_{ij} , there is one and only one connection which preserves g and has torsion equal to T^k_{ij} .

Both curvature and torsion are properties of a connection (Kobayashi and Nomizu 1963). There are in principle an infinity of connections which preserve a given metric g_{ij} . Of all these connections only one, the Levi-Civita connection $\hat{\Gamma}$, has vanishing torsion (a weak version of the Ricci lemma). The others differ from that privileged one precisely by their torsions. The components of the Levi-Civita connection are the well-known Christoffel symbols

$$\hat{\Gamma}^k_{ij} = \frac{1}{2} g^{kr} [\partial_i g_{jr} + \partial_j g_{ir} - \partial_r g_{ij}] \quad (27)$$

The strictly Riemannian curvature \hat{R} will be

$$\hat{R}^r{}_{sij} = \partial_i \hat{\Gamma}^r{}_{sj} - \partial_j \hat{\Gamma}^r{}_{si} + \hat{\Gamma}^r{}_{ki} \hat{\Gamma}^k{}_{sj} - \hat{\Gamma}^r{}_{kj} \hat{\Gamma}^k{}_{si} \quad (28)$$

A connection exhibits torsion in the generic case. Now, given a general connection Γ^k_{ij} preserving a metric g_{ij} , it can always be written in the form

$$\Gamma^k_{ij} = \overset{\circ}{\Gamma}^k_{ij} - K^k_{ij} \quad (29)$$

where K^k_{ij} is its contorsion tensor. Any two connections differ by some tensor, but here metric compatibility gives an extra constraint: contorsion is fixed by the torsion tensor,

$$K^k_{ij} = \frac{1}{2}[T^k_{ij} + T_{ij}^k + T_{ji}^k] \quad (30)$$

This comes from the comparison of two expressions for $\overset{\circ}{\Gamma}$: one obtained by substituting (26) in (27) three times; the other by using (20) in (29). As both T and K are tensors, this relationship holds in any basis. Notice that the decompositions (20) and (29) are not the same. The two last terms in (30) show a symmetric contribution of contorsion to Γ : $K^k_{(ij)} = T_{(ij)}^k$. In consequence,

$$\Gamma^k_{(ij)} = \overset{\circ}{\Gamma}^k_{ij} - T_{(ij)}^k \quad (31)$$

and

$$\Gamma^k_{[ij]} = -K^k_{[ij]} = -T^k_{ij} \quad (32)$$

The property

$$K_{(ki)j} = 0 \quad (33)$$

follows from the fact that $\overset{\circ}{\Gamma}$ satisfies (26) independently.

The presence of torsion changes curvature. Indeed, the total curvature (16) is

$$R^r_{sij} = \overset{\circ}{R}^r_{sij} - M^r_{sij} \quad (34)$$

where

$$\begin{aligned} M^r_{sij} = & \partial_i K^r_{sj} - \partial_j K^r_{si} + \overset{\circ}{\Gamma}^r_{mi} K^n_{sj} + K^r_{ni} \overset{\circ}{\Gamma}^n_{sj} - \overset{\circ}{\Gamma}^r_{nj} K^n_{si} - K^r_{nj} \overset{\circ}{\Gamma}^n_{si} \\ & - K^r_{ni} K^n_{sj} + K^r_{nj} K^n_{si} \end{aligned} \quad (35)$$

We have thus obtained a geometrized version of the Yang–Mills system in Euclidean space. The original $SU(2)$ connection A has been transformed into a linear connection on \mathbf{E}^3 , which will be “felt” by $SU(2)$ nonsinglet particles. Notice that, once a particular dreibein field is used, g_{ij} , $\overset{\circ}{\Gamma}$, and $\overset{\circ}{R}^r_{sij}$ are fixed. The metric being arbitrary, the trivial choice would be a flat host space: $\overset{\circ}{\Gamma} = 0$, $\overset{\circ}{R} = 0$. In that case, the equations reduce to

$$\begin{aligned} M^r_{sij} = & \partial_i K^r_{sj} - \partial_j K^r_{si} - K^r_{ki} K^k_{sj} - K^r_{kj} K^k_{si} \\ & \partial_j M^r_{si}{}^j - K^r_{kj} M^k_{si}{}^j + K^k_{sj} M^r_{ki}{}^j = 0 \end{aligned}$$

These are just the equations (5) and (9) we started from, with the gauge

potential transmuted into a contorsion by trivial dreibeine. Only the indices related to the gauge Lie algebra have been changed into 3-space indices up to now. And, as we treat the new indices (r, s, \dots) on an equal footing with the original, holonomic ambient 3-space indices (i, j, \dots), what we are actually doing is to choose a new basis for the algebra at each point of the ambient space (in the language of the sixties, we are geometrizing the “internal space”). This is better understood if we consider the complete expressions of the algebra-valued differential forms involved. In (14), for example, what we have is

$$\begin{aligned} R &= \frac{1}{2} J_c^b R^c_{bij} dx^i \wedge dx^j = \frac{1}{2} J_c^b h^c_r h^s_b R^r_{sij} dx^i \wedge dx^j \\ &= \frac{1}{2} J_r^s(x) R^r_{sij} dx^i \wedge dx^j \end{aligned}$$

We remain, for the time being, on the original flat space. We shall later discuss the meaning of “geometrizing” the ambient space indices as well.

A few words on the Wu–Yang ambiguity. Two distinct gauge potentials which have the same curvature are called “copies.” It should be said, to begin with, that there are no copies in the relationship between $\mathring{\Gamma}$ and \mathring{R} . There exists always around each point p a system of coordinates in which $\mathring{\Gamma} = 0$ at p , so that the usual expression $R = d\mathring{\Gamma} + \mathring{\Gamma}\mathring{\Gamma}$ reduces to $\mathring{R} = d\mathring{\Gamma}$, which can be integrated to give locally $\mathring{\Gamma}$ in terms of \mathring{R} . This is analogous to the equivalence principle which protects standard general relativity from ambiguity, but holds only for symmetric, torsionless connections. On the other hand, general linear connections exhibit copies in a natural way, as they can, in principle, have the same curvature and different torsions. For example, each solution K^r_{si} (if any) of $M^r_{sij} = 0$ in (35) will lead to a copy of the Levi-Civita connection. Instead of solving this equation, however, it is simpler to take the difference between the two Bianchi identities, which leads to an algebraic condition for the nonexistence of copies (Roskies, 1977; Calvo, 1977). Though powerful general results have been found on the problem (Mostow, 1980; Doria, 1981), there seems to be no simple, systematic, calculating view of the problem.

Suppose now we start with two distinct potentials A and A' and transcribe them into 3-space geometries using the same dreibein field. As g and $\mathring{\Gamma}$ will be the same, their difference will be in their contorsions. This will lead to different curvatures and torsions. If, however, A and A' have the same curvature, only torsion will remain to distinguish them. Thus, copies are “classified” by torsions. This is trivial for linear connections, but the good thing about the geometrization given above is exactly that: we can transfer to gauge potentials, which are connections related to internal groups, some of the properties of the linear, external connections.

Notice also that discussions on the ambiguity are not, in general, concerned with solutions and mostly ignore the dynamic classical equations. Nonsolutions are very important because they appear as off-shell contributions in the quantum case. The geometric formulation has been used to produce examples of continuous sets of copies (Freedman and Khuri, 1994).

We study now cases in which it is possible to choose the metric so as to completely absorb the gauge field.

4. ISOTROPIC OPTICS

Each choice of dreibeine will provide a different transcription into a 3-space geometry. A natural question is whether it is possible to choose them so as to absorb the gauge field entirely in the metric sector alone, dispensing with the torsion field. Is it possible to transmute the gauge field into pure optics? This would mean finding a dreibein field inducing a metric whose Levi-Civita connection $\overset{\circ}{\Gamma}$ coincides with the transcript Γ of A^a_j .

Consider dreibeine of the form

$$h^a_i = \delta^a f(r) \tag{36}$$

where $f(r)$ is any function depending only on the distance r to some fixed origin. The metric they define,

$$g_{ij} = \delta_{ij}[f(r)]^2 \tag{37}$$

has the Levi-Civita connection

$$\overset{\circ}{\Gamma}^k_{ij} = (\delta^k_j x_i + \delta^k_i x_j - \delta_{ij} x^k) \frac{1}{rf} \frac{\partial f}{\partial r} \tag{38}$$

with curvature

$$\begin{aligned} \overset{\circ}{R}^t_{sij} &= \frac{1}{f} \frac{\partial f}{\partial r} \left(\frac{2}{r} + \frac{1}{f} \frac{\partial f}{\partial r} \right) (\delta^t_j \delta_{si} - \delta^t_i \delta_{sj}) \\ &+ \frac{1}{r^2 f} \left[\frac{\partial^2 f}{\partial r^2} - \frac{1}{r} \frac{\partial f}{\partial r} - \frac{2}{f} \left(\frac{\partial f}{\partial r} \right)^2 \right] \\ &\times (\delta^t_j x_i x_s - \delta_{sj} x^t x_i - \delta^t_i x_j x_s + \delta_{si} x^t x_j) \end{aligned} \tag{39}$$

Looking for solutions of the Yang–Mills equations, we take (38) and (39) into (17) and find

$$\frac{\partial^3 f}{\partial r^3} + \frac{1}{rf} \left(\frac{\partial f}{\partial r} \right)^2 - \frac{5}{f} \frac{\partial f}{\partial r} \frac{\partial^2 f}{\partial r^2} + \frac{5}{f^2} \left(\frac{\partial f}{\partial r} \right)^3 = 0 \tag{40}$$

Using equations (15) and (10) in (38), we find the potential

$$A_j^d = -\frac{1}{rf} \frac{\partial f}{\partial r} \epsilon_{jk}^d x^k \quad (41)$$

From (39), (14) and (11) we have the magnetic field

$$B_j^d = \delta_j^d \left[\frac{2}{rf} \frac{\partial f}{\partial r} + \frac{4}{f^2} \left(\frac{\partial f}{\partial r} \right)^2 - \frac{1}{f} \frac{\partial^2 f}{\partial r^2} \right] + \frac{1}{r^2 f} \left[\frac{\partial^2 f}{\partial r^2} - \frac{1}{r} \frac{\partial f}{\partial r} - \frac{2}{f} \left(\frac{\partial f}{\partial r} \right)^2 \right] x_j x^d \quad (42)$$

Any solution of (40) will lead to a solution of (9) given by (41) and (42).

Let us discuss a few particular cases. Consider $f(r) = 1/r^q$, defined in all points of space except $r = 0$. The result of introducing it into (40) is an equation for q ,

$$q(2 - q)(1 - q) = 0 \quad (43)$$

which three obvious solutions. Cases $q = 0$ and $q = 2$ lead to trivial solutions: both the potential and the curvature are zero in the first case, and the second corresponds to a nonvanishing potential with zero curvature. The only nontrivial solution is $q = 1$, $f(r) = 1/r$, the well-known Wu–Yang monopole (Wu and Yang 1975)

$$A_j^a = \epsilon_{jk}^a \frac{x^k}{r^2} \quad (44)$$

$$B_j^a = -\frac{x^a x_j}{r^4} \quad (45)$$

Expressions of the type $\exp(\pm r^q)$, $\exp[1/(1 - qr)]$, $\exp[1/(1 - r^q)]$, and $\exp(\pm qr^2)$ are only real solutions for $q = 0$. Expressions of the type $[r/(1 - qr)]$ and $[r/(1 - qr^2)]$ have only complex solutions for q . The monopole is the only nontrivial solution found. No torsion is necessary in this case. There is a metamorphosis of the gauge field into an isotropic optics with refractive index $n = 1/r$. It has been possible to choose a “host” Riemannian background which entirely incorporates the gauge field. In that case, there can exist no copies (the copy exhibited by Wu and Yang appears in the presence of a source current).

In the example above we have taken a solution of the classical field equation. Solutions or not, general field configurations of the form

$$A_j^a = -\epsilon_{jk}^a \partial^k \ln n(x) \quad (46)$$

are taken by the dreibeine $h_i^a = \delta_i^a n$ into connections which coincide with

the Christoffel symbols of the corresponding metric $g_{ij} = \delta_{ij}n^2$. There is something curious about such cases: a system of coordinates exists in which a symmetric connection vanishes. There is consequently a kind of equivalence principle for this type of gauge field: by a judicious choice of dreibeine, and then of coordinates, the potential (though not the field strength) can be made to vanish.

5. PROBING INTO INTERNAL SPACE

We can use some general characteristics to investigate the “internal geometry” obtained. Geodesics, for example, have a strong mathematical appeal, and are much used in gravitation to describe general qualitative properties of spaces. It is natural to ask whether they have some role here. In the pure-optics case, as the light-ray equation, the geodesic equation does provide an intuitive picture of the system.

From the strictly metric-Riemannian point of view, the geodesic equation for the general case,

$$\frac{dv^i}{ds} + \overset{\circ}{\Gamma}{}^i{}_{jk}v^jv^k = T_{jk}{}^i v^jv^k \tag{47}$$

can be seen as a kind of force law. The right-hand side would vanish for shortest-length curves. As it does not, shortest-length curves are not self-parallel. Notice that the affine parameter s has nothing to do with time, and \mathbf{v} is only a unit vector tangent to the curve. No physical particle is expected to follow such a path.

On the other hand, parallel transport is taken into parallel transport by the geometrizing transcription. A test particle in a gauge field is described by (i) its spacetime coordinates and (ii) an “internal” vector $\mathbf{I} = \{I_a\}$ giving its state in isotopic space. The corresponding dynamic equations (Wong, 1970; Drechsler and Rosenblum, 1981) are (i) the generalized Lorentz force law, which for a unit mass reads

$$\frac{d^2x^\mu}{d\tau^2} = I_a F^{a\mu\nu} \frac{dx_\nu}{d\tau} \tag{48}$$

and (ii) the so-called charge-precession equation,

$$\frac{d\mathbf{I}}{d\tau} + \mathbf{A}_\mu \times \mathbf{I} \frac{dx^\mu}{d\tau} = 0 \tag{49}$$

The latter says that internal motion is a parallel-transport and a precession, as \mathbf{I}^2 is conserved. Its transcription,

$$\frac{dI^i}{ds} + \Gamma_{jk}^i I^j v^k = \frac{DI^i}{Ds} = 0$$

says that the transcript of \mathbf{I} precesses parallel-transported by Γ along the transcribed curve. If we take for \mathbf{I} a current $\mathbf{I} = k\mathbf{v}$, it gives just the geodesic equation. As to the Lorentz law, it takes the form

$$\frac{dv^i}{ds} - \frac{1}{2} I^s R^r_{\ s}{}^i{}_{\ j} v^j = 0 \quad (50)$$

This expression, with a velocity–curvature coupling, is more akin to the Jacobi than to the geodesic equation. It implies the conservation of \mathbf{v}^2 . Combining the geodesic equation with the charge-precession equation, we find

$$\frac{d(I_i v^i)}{ds} = 0$$

Thus, \mathbf{I} keeps constant its component along a geodesic.

6. FULL GEOMETRIZATION

In all we have done previously, the original, ambient space indices have been preserved. Only algebra-related indices have been “geometrized.” This has the advantage of simplicity, as all the expressions are written in the initial holonomic basis of ambient space. That space remains what it was, the Euclidean 3-dimensional flat space, and solutions eventually found will be solutions in flat space. We can now proceed to a complete transmutation into curved space, including the ambient space. We identify the two original Euclidean flat spaces and use the dreibeine to pass entirely into the new space. This will lead to more involved expressions, as everything will appear written in the anholonomic basis defined by the dreibeine. It will have, however, a double merit: we shall be able to speak really of optics, and new solutions will turn up. Indices of both spaces, internal and ambient, become of the same kind and can be mixed, as they are in general relativity. A typical example of such mixing is the so-called cyclic identity for the Riemann tensor, which comes from (24) when $T^a = 0$.

To work in an anholonomic basis is, on the other hand, a troublesome task. Thus, after performing the complete transposition, it will better to choose a coordinate basis again. Coordinates of a 3-space are functions with values in \mathbf{E}^3 . It is particularly interesting to choose the original ambient space coordinates as coordinates of the new space because, except for the terms involving derivatives, all the above expressions remain formally the same. Thus, (17) becomes

$$\frac{1}{\sqrt{|g|}} \partial_j [\sqrt{|g|} R^r_{si}{}^j] + \Gamma^r_{kj} R^k_{si}{}^j - \Gamma^k_{sj} R^r_{si}{}^j = 0 \tag{51}$$

Notice that this equation is the transcription of the static Ampère equation (9), a particular case of the Yang–Mills equations in 4-dimensional spacetime. Despite its aspect, there is *a priori* no reason for it to have any special significance by itself. It so happens, however, that (51) is precisely the sourceless Yang–Mills equation on the 3-dimensional curved space with metric g . This equation is defined (Nowakowski and Trautman, 1978) on any space as the natural generalization of the flat case: the covariant coderivative of the curvature equals zero (Aldrovandi and Pereira, 1995):

$$*^{-1}d * R + *^{-1}[\Gamma, *R] = 0 \tag{52}$$

where $*$ represents the Hodge star operator. Some attention must be paid to the signature of the “host” metric, but in any sourceless case the equation has the component form (51).

Here comes its main interest: it is known (Nowakowski and Trautman, 1978; Harnad *et al.*, 1980) that the sourceless Yang–Mills equation on a symmetric homogeneous space is solved by the corresponding canonical connection. In consequence, any 3-dimensional homogeneous symmetric space will provide a solution for (51). These connections are torsionless, so that we come back to pure optics. Furthermore, they have constant scalar curvature. Purely Riemannian spaces of constant curvature are not so many: they are those hosting the highest possible number of Killing vectors. Given the metric signature and the value of the scalar curvature R , there is only one such “maximally symmetric” space (Weinberg, 1972), provided torsion is absent. A negative constant total scalar curvature would establish the space as a hyperbolic space. Thus, once a complete transmutation is performed, each symmetric homogeneous space will provide an optics which solves the Yang–Mills equation. The simplest 3-dimensional cases are the sphere S^3 and the hyperbolic spaces. These would be the cases of static Yang–Mills equation in Friedmann (respectively closed and open) model universes.

Consider, to start with, the hypersphere S^3 in \mathbf{E}^4 , given in Cartesian coordinates $\{\xi^\mu\}$ by $\sum_{\mu=1}^4 (\xi^\mu)^2 = (\xi^4)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 = 1$. We can project it stereographically from the point $\xi^4 = +1$ (its “north pole”) into the hyperplane \mathbf{E}^3 tangent at the point $\xi^4 = -1$ (the “south pole”). This will provide every point of the hypersphere (except the north pole) with coordinates $x^k = 2\xi^k/(1 - \xi^4)$ on \mathbf{R}^3 . It is a direct adaptation of the Riemannian metric of S^3 on the Euclidean space, with the north pole corresponding to all the points at infinity. Introducing $r^2 = \sum_{i=1}^3 (x^i)^2$ and calculating the line element $ds^2 = \sum_{\mu} (\xi^\mu)^2$ in these stereographic coordinates, we obtain the

spherical metric $ds^2 = g_{ij} dx^i dx^j$, where $g_{ij} = n^2(x) \delta_{ij}$, with $n = \frac{1}{2}(1 - \xi^4) = 1/(1 + r^2)$. This case is well known in geometrical optics, where, with n the refraction index, it leads to the perfectly focusing Maxwell fisheye (Luneburg, 1966). It does not lead to any bounding in space, as the sphere is taken onto the whole of \mathbf{R}^3 . It is a conformally flat space, as the new metric is at each point proportional to the Euclidean metric.

Take now a hyperbolic space in \mathbf{E}^4 , given by $(\xi^4)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 = 1$. It consists of two branches, each one a Lobachevski space. The points $\xi^4 = +1$ and $\xi^4 = -1$ are now the lowest point of the upper branch and the highest point of the lower branch. The stereographic projection leads now to a metric $g_{ij} = n^2(x)\delta_{ij}$ with the refraction index $n = 1/(1 - r^2)$. In other words, given a hyperbolic metric on \mathbf{R}^3 , it is always possible to find a coordinate system $\{x^i\}$ in terms of which the metric is $g_{ij} = n^2(x)\delta_{ij}$, with n as above and $r^2 = \sum_{i=1}^3 (x^i)^2$. Higher dimensional analogues are the anti-de Sitter spaces, which may exhibit properties analogous to perfect focusing (Hawking and Ellis, 1973).

There are two kinds of hyperbolic space, the one-sheeted and the two-sheeted. Now, it is a well-known fact that in the two-sheeted case the above stereographic coordinates divide \mathbf{R}^3 into two parts, one for each branch of the hyperbolic space (Aldrovandi and Pereira, 1995). One of them is a ball, a Poincaré space, the interior ($r < 4$) of a sphere S^2 (where $r = 4$), the other ($r > 4$) its complement in \mathbf{R}^3 . The geodesics are easily computed, and better suited to get some intuition about what happens, showing an “optics” with some great differences with respect to Maxwell’s fisheye. This shows a “confining” behavior, which is a global effect of the hyperbolic geometry. Locally, one could be misled by intuition, as neighboring geodesics tend to approach each other in the spherical case, thereby simulating an attraction, and to separate from each other in the hyperbolic case (Arnold, 1978). The bounding sphere S^2 itself is a singular region, corresponding to the infinite regions of both branches. It plays the role of a “natural” bag. Of course, there is no reason to believe that test particles will follow geodesics, but actually all continuous paths starting inside the region are trapped within it. There is another point: the metric g_{ij} becomes infinite on the bounding sphere, so does the magnetic field B , and, consequently, the energy density in (8). Space is in this way divided into two regions separated by a barrier on which the energy density diverges. This “confinement” remains, of course, of academic interest, because it only occurs when the ambient space is hyperbolic.

Summing up: A Yang–Mills field can, under certain conditions, be described as an optical medium on 3-space. This fact leads to an alternative to the usual potential picture as a source of ideas and physical intuition.

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REFERENCES

- Aldrovandi, R., and Pereira, J. G. (1995). *An Introduction to Geometrical Physics*, World Scientific, Singapore.
- Arnold, V. I. (1978). *Mathematical Methods of Classical Mechanics*, Springer, New York, Appendix.
- Calvo, M. (1977). *Phys. Rev. D* **15**, 1733.
- Chandrasekhar, S. (1992). *The Mathematical Theory of Black Holes*, Oxford University Press.
- Doria, F. A. (1981). *Commun. Math. Phys.* **79**, 435.
- Drechsler, W., and Rosenblum, A. (1981). *Phys. Lett.* **106B**, 81.
- Faddeev, L. D., and Slavnov, A. A. (1978). *Gauge Fields. Introduction to Quantum Theory*, Benjamin/Cummings, Reading, Massachusetts.
- Feynman, R. P. (1977). In *Weak and Electromagnetic Interactions at High Energy*, North-Holland, Amsterdam, ff. pp. 191.
- Freedman, D. Z., and Khuri, R. R. (1994). *Phys. Lett. B* **329**, 263.
- Freedman, D. Z., Haagensen, P. E., Johnson, K., and Latorre, J. I. (1993). MIT preprint CTP 2238.
- Greub, W. H., Halperin, S., and Vanstone, R. (1972). *Connections, Curvature and Cohomology*, Volume II: *Lie Groups, Principal Bundles and Characteristic Classes*, Academic Press, New York, p. 344.
- Guillemin, V., and Sternberg, S. (1977). *Geometric Asymptotics*, AMS, Providence, Rhode Island.
- Haagensen, P. E. (1993). Lecture at the XIII Particles and Nuclei International Conference, Perugia, Italy, June–July 1993; Barcelona preprint UB-ECM-PF 93/16.
- Harnad, J., Tafel, J., and Shnider, S. (1980). *J. Math. Phys.* **21**, 2236.
- Hawking, S. W., and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-Time*, Cambridge University Press.
- Itzykson, C., and Zuber, J.-B. (1980). *Quantum Field Theory*, McGraw-Hill, New York.
- Jackiw, R. (1980). *Rev. Mod. Phys.* **52**, 661.
- Johnson, K. (1992). In *QCD—20 Years Later*, Aachen.
- Kobayashi, S., and Nomizu, K. (1963). *Foundations of Differential Geometry*, Vol. 1, Interscience, New York.
- Lunenburg, R. K., (1966). *Mathematical Theory of Optics*, University of California Press, Berkeley.
- Lunev, F. A. (1992). *Phys. Lett. B* **295**, 99.
- Mostow, M. A. (1980). *Commun. Math. Phys.* **78**, 137.
- Nowakowski, J., and Trautman, A. (1978). *J. Math. Phys.* **19**, 1100.
- Ramond, P. (1981). *Field Theory: A Modern Primer*, Benjamin/Cummings, Reading, Massachusetts.
- Roskies, R. (1977). *Phys. Rev. D* **15**, 1731.
- Weinberg, S. (1972). *Gravitation and Cosmology*, Wiley, New York, esp. Chapter XIII.
- Wong, S. K. (1970). *Nuovo Cimento* **65A**, 689.
- Wu, T. T., and Yang, C. N. (1975). *Phys. Rev. D* **12**, 3843, 3845.